DETERMINATION OF $H^*(BO(k,\cdots,\infty), Z_2)$ AND $H^*(BU(k,\cdots,\infty),Z_2)$

BY

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Introduction. For any space X, we denote by $X(k,\cdots,\infty)$ the total space of the $(k-1)$-connective fibering over X (Hu [4]). This gives fiberings

$$X(k,\cdots,\infty)\to X(k-1,\cdots,\infty)\to K(\pi_{k-1}(X),k-1)$$

and

$$K(\pi_{k-1}(X),k-2)\to X(k,\cdots,\infty)\to X(k-1,\cdots,\infty).$$

The purpose of this paper is to calculate the cohomology with coefficients in Z_2 (Z_2 will be used for all coefficients) for the spaces obtained in this way from the universal base spaces of the infinite orthogonal and unitary groups, BO and BU, respectively. For BU, Adams [1] determined these groups in the stable range.

We shall show that if $k\equiv 0,1,2,4$ (modulo 8), then

$$H^*(BO(k,\cdots,\infty))\cong H^*(K(\pi_k(BO),k))/I(Q_k i_k)\otimes Z_2[\theta_i\,|\,L(i)>\phi(0,k)]$$

where

$$Q_k=\begin{cases} Sq^2 \text{ if } k\equiv 0,1\ (8),\\ Sq^3 \text{ if } k\equiv 2\ (8),\\ Sq^5 \text{ if } k\equiv 4\ (8),\end{cases}$$

$I(Q_k i_k)$ denotes the ideal generated by $Q_k i_k$, $\phi(0,k)$ denotes the number of integers s such that $0<s\leqq k$ and $s\equiv 0,1,2,$ or 4 (modulo 8), $L(i)$ is one more than the number of ones in the dyadic expansion of $i-1$, and θ_i are classes in $H^*(BO)$ congruent to w_i modulo decomposable elements. Similarly, we shall show that

$$H^*(BU(2p,\cdots,\infty))\cong H^*(K(Z,2p))/I(Sq^3 i_{2p})\otimes Z_2[\theta_{2i}\,|\,L(2i)>p+1].$$

The proofs of these results and this paper can be outlined as follows. In §1 we construct a new system of generators, the θ_i, for $H^*(BO)$ and $H^*(BU)$ which are closely related to the action of the Steenrod algebras on these groups. In §2, we make a partial determination of the groups $H^*(K(\pi,n))/I$ which were mentioned above. These determinations are made by first analyzing the stable

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case in the Steenrod algebra and then reducing to the unstable case. In §3, we compute the spectral sequences of the fibrations given above, making use of the partially determined groups $H^*(K(\pi,n))/I$, to calculate the desired groups $H^*(X(k,\cdots,\infty))$ inductively. Finally in §4, we apply our calculations to obtain some results in cobordism theory. More specifically, we show that a k-parallelizable differentiable manifold of dimension less than $2^{\phi(0,k)+1}$ is cobordic to zero in the unoriented sense. If in addition, the manifold is assumed to be weakly complex, this dimension can be raised to $2^{[k/2]+2}$.

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1. $H^*(BO)$ **and** $H^*(BU)$. As is well known (Milnor [5]), the cohomolgy of BO is a polynomial algebra over Z_2 with generators $w_i \in H^i(BO)$, $i \geqq 0$, $w_0 = 1$, and the action of the Steenrod algebra is given by the formulae of Wu [7],

$$Sq^i w_j = \sum_{t=0}^{i} \binom{j-i-1+t}{t} w_{i-t} w_{j+t}$$

for $i < j$. Further, the cohomology of BU is the quotient of that of BO by the ideal generated by all odd dimensional elements. Thus $H^*(BU)$ is the Z_2 polynomial algebra on the classes w_{2i}.

DEFINITION. An admissible sequence

$$I = (2^{s_q}k_q, \cdots, k_q, 2^{s_{q-1}}k_{q-1}, \cdots, \cdots, k_2, 2^{s_1}k_1, \cdots, k_1)$$

with $k_i > 2^{s_{i-1}+1}k_{i-1}$, will be called a Θ^m-sequence if

(a) I is empty, or

(b) there exist integers $0 \leqq r_q < \cdots < r_1 < m$ such that

$$k_1 = 2^m - 2^{r_1}, \text{ and}$$

$$k_i = 2^{r_{i-1}} - 2^{r_i} + 2^{s_{i-1}+1}k_{i-1}.$$

The excess of a Θ^m-sequence of type (b) is $2^m - 2^{r_q} < 2^m$.

DEFINITION. For any integer j, define $L(j)$ to be one plus the number of one's in the dyadic expansion of $j-1$. ($L(0) = \infty$).

For each integer j such that $L(j+2^m) = m+1$, there is a unique Θ^m-sequence of degree j. In fact, if

$$i - 1 = 2^{p_1} + \cdots + 2^{p_1+r_1} + 2^{p_2} + \cdots + 2^{p_s} + + 2^{p_s+r_s},$$

with $q + \sum_1^s r_i = m$, $p_i > p_{i-1} + r_{i-1} + 1$, let

$$i_1 = \begin{cases} \dfrac{i-1}{2} & \text{if } p_1 > 0, \\ 2^{p_2-1} + \cdots + 2^{p_2+r_2-1} + \cdots + 2^{p_s-1} + \cdots + 2^{p_s+r_s-1} & \text{if } p_1 = 0. \end{cases}$$

$i - i_1 - 1$ also has m one's in its dyadic expansion, and iteration of this process gives a Θ^m-sequence $(i_1, \cdots, i_r)$ of degree $i - 2^m$.

DEFINITION. For each integer i, if $L(i) = m + 1$, let I_i denote the unique Θ^m-sequence of degree $i - 2^m$, and set $\theta_i = S_q^{I_i} w_2^m \in H^i(BO)$. (Let $\theta_0 = 1$.)

PROPOSITION. *$\theta_i \equiv w_i$ modulo decomposable elements.*

Proof. By the Wu formulae,

$$Sq^i w_j \equiv \binom{j-1}{i} w_{i+j},$$

so if $I = (i_1, \cdots, i_r)$,

$$Sq^I w_{2^m} \equiv \binom{2^m + i_2 + \cdots + i_r - 1}{i_1} \cdots \binom{2^m + i_r - 1}{i_{r-1}} \binom{2^m - 1}{i_r} w_{2^m + \deg I}.$$

If I is a Θ^m-sequence, it is immediate that all of these binomial coefficients are 1 (modulo 2).

COROLLARY. *$H^*(BO)$ is the Z_2 polynomial algebra on the classes θ_i, and $H^*(BU)$ is the Z_2 polynomial algebra on the classes Θ_{2i} (or really, their images).*

THEOREM. (a) *The polynomial ideal in $H^*(BO)$ generated by θ_i such that $L(i) \leqq p + 1$ is an ideal under the action of the Steenrod algebra.*

(b) *$H^*(K(Z_2, 2p))/I(Sq^1 i_{2p}, \cdots, Sq^{2^{p-2}} i_{2p})$ is isomorphic to the Z_2 polynomial algebra on the $Sq^I i_{2p}$ with I a Θ^p-sequence.*

Proof. We shall induct on p. For $p = 0$, (b) says simply that $H^*(K(Z_2, 1))$ is the Z_2 polynomial algebra on i_1. For $p = 0, L(i) \leqq 1$ implies $i = 1$. Then $\theta_1 = w_1$, and the polynomial ideal generated by w_1 is all $\alpha \cdot w_1$. Since $Sq^i(\alpha \cdot w_1) = (Sq^i \alpha + Sq^{i-1} \alpha \cdot w_1) \cdot w_1$, this ideal is closed under the Steenrod algebra.

Now suppose (a) and (b) are true if $p < m$. If the $Sq^I i_{2^m}$ with I a Θ^m-sequence generate $H^*(K(Z_2, 2^m))/I(Sq^1 i_{2^m}, \cdots, Sq^{2^{m-2}} i_{2^m})$, (a) and (b) hold for $p = m$. More precisely, let $f: BO \to K(Z_2, 2^m)$ with $f^*(i_{2^m}) = w_{2^m}$. Then

$$H^*(K(Z_2, 2^m)) \xrightarrow{f^*} H^*(BO) \to H^*(BO)/I(\theta_i \,|\, L(i) \leqq m) = B$$

is by $(a)_{m-1}$ a homomorphism over the Steenrod algebra. Since

$$B \cong Z_2[\theta_i | L(i) \geqq m + 1],$$

B has no elements of dimension n with $2^m + 2^{m-1} > n > 2^m$, so this induces

$$A = H^*(K(Z_2,2^m))\ I(Sq^1 i_{2^m},\cdots,Sq^{2^{m-2}} i_{2^m}) \overset{g}{\to} B.$$

Since A is generated by the Θ^m-sequences, the image of g is contained in $Z_2[\theta_i \mid L(i) = m+1]$. Further, there is one θ_i for each generator, so $\theta_i \overset{h}{\to} Sq^{I_i} i_{2^m}$ gives

$$Z_2[\theta_i \mid L(i) = m+1] \overset{h}{\to} A \overset{g}{\to} Z_2[\theta_i \mid L(i) = m+1].$$

with h epic and $g \circ h$ an isomorphism. Thus h and g are isomorphisms. Since h is an isomorphism, $(b)_m$ is true. Since g is an epimorphism, the polynomial ideal generated by $\{\theta_i \mid L(i) = m+1\}$ is an ideal over the Steenrod algebra in B, so its inverse image, the polynomial ideal generated by $\{\theta_i \mid L(i) \leqq m+1\}$ in $H^*(BO)$, is an ideal over the Steenrod algebra.

Thus it suffices to show A is generated by all $Sq^I i_{2^m}$, with I a Θ^m-sequence. Let $C = H^*(K(Z_2,2^m))$, $K = I(Sq^1 i_{2^m},\cdots,Sq^{2^{m-2}} i_{2^m})$. Since C is generated by all $Sq^I i_{2^m}$ with I admissible and the excess of $I = e(I)$ less than 2^m, it suffices to show that if I is admissible, $e(I) < 2^m$, and I is not a Θ^m-sequence, then $Sq^I i_{2^m} \in K$. If $I = (i_1,\cdots,i_r)$, r will be called the length of I, and we shall prove this by induction on the length of I.

If length $I = 0$, I is the empty sequence, which is a Θ^m-sequence.

If length $I = 1$, $I = (i)$ has excess less than 2^m only if $i < 2^m$. If I is not a Θ^m-sequence, $i \neq 2^m - 2^n$, $n < m$. If $i < 2^{m-1}$, then $Sq^I i_{2^m} \in K$ since the Sq^{2^α} generate the Steenrod algebra. Thus $i = 2^{m-1} + \cdots + 2^{m-p} + b$, with $0 < b < 2^{m-p-1}$. The Adem relations [3] give

$$Sq^{2^{m-2}+\cdots+2^{m-p-1}} Sq^{2^{m-2}+\cdots+2^{m-p-1}+b} = \sum_{t=0}^{[2^{m-3}+\cdots+2^{m-p-2}]} \binom{2^{m-2}+\cdots+2^{m-p-1}+b-1-t}{2^{m-2}+\cdots+2^{m-p-1}-2t} Sq^{i-t} Sq^t.$$

$b - 1 < 2^{m-p-1} - 1 = 1 + \cdots + 2^{m-p-2}$, so

$$\binom{2^{m-2}+\cdots+2^{m-p-1}+b-1}{2^{m-2}+\cdots+2^{m-p-1}} \equiv 1 \quad \text{(modulo 2)},$$

and

$$Sq^i i_{2^m} = Sq^{2^{m-2}+\cdots+2^{m-p-1}} Sq^{2^{m-2}+\cdots+2^{m-p-1}+b} i_{2^m} + \sum Sq^{i-t} Sq^t i_{2^m},$$

the sum being over some t with $1 \leqq t \leqq [2^{m-3} + \cdots + 2^{m-p-2}] < 2^{m-1}$. Thus $Sq^t i_{2^m} \in K$ for all such t, and $2^{m-2} + \cdots + 2^{m-p-1} + b < 2^{m-1}$, so

$$Sq^{2^{m-2}+\cdots+2^{m-p-1}+b}\, i_{2^m} \in K.$$

Hence $Sq^i i_{2^m} \in K$.

Thus inductively, if I is admissible, $e(I) < 2^m$, length $I \leqq p$ $(p \geqq 1)$ and I is not a Θ^m-sequence, then $Sq^I i_{2m} \in K$.

Now let $I = (i_1, \cdots, i_{p+1})$ be admissible, $e(I) < 2^m$, I not a Θ^m-sequence. Let $J = (i_2, \cdots, i_{p+1})$, $\tilde{J} = (i_3, \cdots, i_{p+1})$. J is admissible, $e(J) < 2^m$, and J has length p, so if J is not a Θ^m-sequence, $Sq^I i_{2m} = Sq^{i_1} Sq^J i_{2m} \in K$. Thus we may assume J is a Θ^m-sequence.

Let $i = \dim(Sq^J i_{2m})$. Then $i - 1 = 2^{p_1} + \cdots + 2^{p_1 + r_1} + \cdots + 2^{p_q} + \cdots + 2^{p_q + r_q}$, $p_i > p_{i-1} + r_{i-1} + 1$, $q + \sum_1^q r_i = m$.

If $p_1 > 0$, $i_2 = (i-1)/2$. Since I is not a Θ^m-sequence and is admissible, $i_1 > 2i_2 = i - 1$, and $e(I) = 2i_1 - \deg I = 2i_1 - (i_1 + i - 2^m) = 2^m + i_1 - i < 2^m$ implies $i_1 < i$. This is impossible, so $p_1 = 0$.

Thus $i - 1 = 1 + \cdots + 2^{r_1} + 2^{p_2} + \cdots = 1 + \cdots + 2^{r_1} + 2i_2$, and by the above $2i_2 < i_1 < i$. Let $i_1 = 2i_2 + a$, $a = 2^{s_1} + \cdots + 2^{s_\beta}$, $0 \leqq s_1 < \cdots < s_\beta \leqq r_1$. Since I is not a Θ^m-sequence, $s_\beta < r_1$ or $a = 2^{s_1} + \cdots + 2^{s_t} + 2^{r_1 - s} + \cdots + 2^{r_1}$, $r_1 - s > s_t + 1$, $s_1 < \cdots < s_t$.

Consider $Sq^{2i_2} Sq^{i_2 + a} Sq^{\tilde{J}} i_{2m}$. $Q = (i_2 + a, i_3, \cdots, i_{p+1})$ is admissible, has length p, and

$$\begin{aligned} e(Q) &= 2(i_2 + a) - (i_2 + a + i_3 + \cdots + i_{p+1}) \\ &= 2i_2 - (i_2 + \cdots + i_{p+1}) + a \\ &= e(J) + a, \\ e(I) &= 2i_1 - (i_1 + \cdots + i_{p+1}) \\ &= i_1 - (i_2 + \cdots + i_{p+1}) \\ &= 2i_2 + a - (i_2 + \cdots + i_{p+1}) \\ &= e(J) + a, \end{aligned}$$

so $e(Q) < 2^m$. Q is not a Θ^m-sequence, for

(1) if $i_2 = 2i_3 + 2^p - 2^{r_1 + 1} = 2i_3 + 2^{p-1} + \cdots + 2^{r_1 + 1}$

$$\begin{aligned} i_2 + a &= 2i_3 + 2^{p-1} + \cdots + 2^{r_1+1} + 2^{r_1} + \cdots + 2^{r_1 - s} + 2^{s_t} + \cdots + 2^{s_1} \quad \text{or} \\ &= 2i_3 + 2^{p-1} + \cdots + 2^{r_1+1} + 2^{s_\beta} + \cdots + 2^{s_1} \quad \text{if } s_\beta < r_1, \end{aligned}$$

and so in either case there is a gap in the sum of powers of two.

(2) If $i_2 = 2i_3$, then $i_2 + a \neq 2i_3$, $2i_3 + 2^{r_1 + 1} - 2^q$.

Thus $Sq^{2i_2} Sq^{i_2 + a} Sq^{\tilde{J}} i_{2m} \in K$.

Now $2i_2 < 2(i_2 + a)$, so the Adem relations give

$$\sum_{t=0}^{i_2} \binom{i_2 + a - t - 1}{2i_2 - 2t} Sq^{3i_2 + a - t} Sq^t Sq^{\tilde{J}} i_{2m} \in K$$

or, since

$$\binom{i_2+a-t-1}{2i_2-2t} = 0\,,$$

unless $i_2+a-t-1 \geqq 2i_2-2t$, $t \geqq i_2-a+1$ and

$$\binom{i_2+a-1-i_2}{2i_2-2i_2} = \binom{a-1}{0} = 1\,,$$

$$\begin{aligned} Sq^I i_{2m} &= Sq^{2i_2+a}Sq^{i_2}Sq^{\tilde{J}}i_2, \\ &\equiv \sum_{t=i_2-a+1}^{i_2-1}\binom{i_2+a-t-1}{2i_2-2t}Sq^{3i_2+a-t}Sq^tSq^{\tilde{J}_1}i_{2m} \qquad \text{(modulo } K\text{)}. \end{aligned}$$

Now consider $(t,\tilde{J}) = (t, i_3, \cdots, i_{p+1})$. Let $t = i_2 - z$, $1 \leqq z \leqq a-1$. Then $(t,\tilde{J}) = \sum J_\alpha$, with J_α admissible. By the Adem relations, length of $J_\alpha \leqq$ length of $(t,\tilde{J}) = p$. If $e(J_\alpha) > 2^m$, $Sq^{J_\alpha}i_{2m} = 0$. If $e(J_\alpha) = 2^m$, $Sq^{J_\alpha}i_{2m} = (Sq^{L_\alpha}i_{2m})^{2^k}$, where L_α is admissible, length $L_\alpha <$ length J_α, and $e(L_\alpha) < 2^m$. Thus $Sq^tSq^{\tilde{J}}i_{2m} \equiv \sum Sq^{\bar{J}_\beta}i_{2m} + \sum (Sq^{\hat{J}_\beta}i_{2m})^{2^k}$ (modulo K), where $\bar{J}_\beta, \hat{J}_\beta$ are Θ^m-sequences.

Now $\dim(Sq^tSq^{\tilde{J}}i_{2m}) - 1 = i - 1 - z = 1 + \cdots + 2^{r_1} + (2^{p_2} + \cdots + 2^{p_q+r_q}) - z$ with $1 \leqq z \leqq a-1 < a < 1 + \cdots + 2^{r_1}$. Thus z cancels some powers of 2 from the part $1 + \cdots + 2^{r_1}$, and hence $L(\dim(Sq^tSq^{\tilde{J}}i_{2m})) < m+1$.

Further, if J_α, L_α are Θ^m-sequences, $L(\dim(Sq^{J_\alpha}i_{2m})) = m+1$, and if

$$\begin{aligned} \dim(Sq^{L_\alpha}i_{2m}) - 1 &= 2^{t_1} + \cdots + 2^{t_m} \qquad t_1 < \cdots < t_m, \\ \dim((Sq^{L_\alpha}i_{2m})^{2^k}) &= 2^k(1 + 2^{t_1} + \cdots + 2^{t_m}) \\ &= 2^k + 2^{k+t_1} + \cdots + 2^{k+t_m}\,, \end{aligned}$$

so

$$\dim((Sq^{L_\alpha}i_{2m})^{2k}) - 1 = 1 + \cdots + 2^{k-1} + 2^{k+t_1} + \cdots + 2^{k+t_m}$$

or

$$L(\dim((Sq^{L_\alpha}i_{2m})^{2^k})) = 1 + m + k > 1 + m.$$

Thus since both sides of

$$Sq^tSq^{\tilde{J}}i_{2m} \equiv \sum Sq^{\bar{J}_\beta}i_{2m} + \sum (Sq^{\hat{J}_\beta}i_{2m})^{2^k}$$

must have the same value of $L\cdot\dim$, we have $Sq^tSq^{\tilde{J}}i_{2m} \equiv 0$ (modulo K) for all in the given range, and thus

$$Sq^I i_{2m} \in K.$$

2. $H^*(K(\pi,n))/I$. In this section, we will determine part of the structure of the groups

$$H^*(K(Z,2p))/I(Sq^3 i_{2p}),$$

$$H^*(K(Z,8k))/I(Sq^2 i_{8k}),\ H^*(K(Z_2,8k+1))/I(Sq^2 i_{8k+1}),$$

$$H^*(K(Z_2,8k+2))/I(Sq^3 i_{8k+2}),$$

and

$$H^*(K(Z,8k+4))/I(Sq^5 i_{8k+4}),$$

since these groups form the building blocks for $H^*(BO)$ and $H^*(BU)$. Our main tool will be the exact sequences of Toda [6],

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\tilde{S}q^2} & \mathscr{A}/\mathscr{A}Sq^1 \\ \tilde{S}q^2 \uparrow & & \downarrow \tilde{S}q^5 \\ \mathscr{A} & \xleftarrow{\tilde{S}q^3} & \mathscr{A}/\mathscr{A}Sq^1 \end{array}$$

and

$$\mathscr{A}/\mathscr{A}Sq^1 \xrightarrow{\tilde{S}q^3} \mathscr{A}/\mathscr{A}Sq^1 \xrightarrow{\tilde{S}q^3} \mathscr{A}/\mathscr{A}Sq^1,$$

where $\mathscr{A}$ denotes the Steenrod algebra, and the mappings are given by right multiplication by the given elements.

In the following we shall write

$$I = (i_1,\cdots,i_r) = [i_1 - 2i_2,\cdots,i_{r-1} - 2i_r, i_r].$$

PROPOSITION. *$\mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^3$ has a basis consisting of elements of the form $I + \sum J$, $J > I$ in lexicographic order; I,J being admissible sequences; with $I = (I_0, I_1)$ where*

(1) I_1 is empty, or

$$I_1 = \begin{cases} [0,\cdots,0,2,0,\cdots,0,2,0,\cdots,0,2,0,\cdots,\cdots,0,2] = A \\ [0,\cdots,0,4,0,\cdots,0,2,0,\cdots,0,2,0,\cdots,\cdots,0,2] = B \\ [0,\cdots,0,4] = C, \end{cases}$$

and

(2) I_0 is empty, or $i_r(I_0) \geqq 2i_1(I_1) + N(I_1)$, where $N(I_1) = 6$ if I_1 is empty or has type A, and $N(I_1) = 2$ if I_1 has type B or C.

Further, the map $\mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^3 \to^{\tilde{S}q^3} \mathscr{A}/\mathscr{A}Sq^1$ takes $I + \sum J \to I' + \sum J'$, where $J' > I'$ in lexicographic order, and $I' = (I_0, I_{1'})$, with

(a) $I_1' = (3)$ *if I_1 is empty,*

(b) $I_1' = [5,0,\cdots,0,2,0,\cdots,2,0,\cdots,2]$ *if* $I_1 = [2,0,\cdots,0,2,0,\cdots,2,0,\cdots,2]$,

(c) $I_1' = [3,0,\cdots,0,2,0,\cdots,2,0,\cdots,2]$ *if* $I_1 = [0,\cdots,0,2,0,\cdots,2,0,\cdots,2]$,

(d) $I_1' = [1,0,\cdots,0,4,0,\cdots,2,0,\cdots,2]$ *if* $I_1 = [0,\cdots,0,4,2,0,\cdots,2,0,\cdots,2]$,

(e) $I_1' = [1,0,\cdots,0,2,0,\cdots,0,2,\cdots,2]$ *if* $I_1 = [0,\cdots,0,4,0,\cdots,0,2,\cdots,2]$, *and*

(f) $I_1' = [1,0,\cdots,0,2]$ *if* $I_1 = [0,\cdots,0,4]$.

Proof. We will say that an element $\alpha = I + \sum J$, $J > I$, has rank k if $I = [0,\cdots,0,2,0,\cdots,0,2,0,\cdots,\cdots,0,2]$ with k 2's or $I = [\cdots,a,0,\cdots,0,2,0,\cdots,0,2,0,\cdots 0,2]$ with $a \neq 0$, 2 and with k 2's.

The principal assertion of the proposition is that there exist elements $\alpha_k = (2^{k+1} - 2, \cdots, 2^2 - 2) + \sum J$, $J > (2^{k+1} - 2, \cdots, 2^2 - 2)$, such that $\widetilde{Sq}^3(\alpha_k) = (2^{k+1} + 1, \alpha_{k-1})$. We take $\alpha_0 = (\phi)$, $\alpha_1 = (2)$, $\alpha_2 = (6,2) + (8)$. Now we suppose α_k exists for $k \leqq p$.

Now

$$\alpha_p = [\underbrace{2,\cdots,2}_{p}] + \cdots \rightarrow (2^{p+1} + 1, \alpha_{p-1}) = [5,\underbrace{2,\cdots,2}_{p-1}] + \cdots.$$

From α_p, we will construct all other elements of rank p in our basis, and construct their images.

First consider (b)

$$[2,0,\cdots,0,2,\cdots,2] \rightarrow [5,0,\cdots,0,2,\cdots,2].$$

In the Adem formulae,

$$Sq^{2p}Sq^{p+2} = \sum_0^p \binom{p+1-t}{2p-2t} Sq^{3p+2-} Sq^t,$$

$$= Sq^{2p+2}Sq^p + \text{terms larger than } (2p+2,p),$$

and similarly

$$Sq^{2p}Sq^{p+5} = Sq^{2p+5}Sq^p + \text{terms larger than } (2p+5,p),$$

and

$$(8i, 4i+4+a, 2i+2, i) = (8i+4+a, 4i, 2i+2, i) + \cdots,$$

$$= (8i+4+a, 4i+2, 2i, i) + \cdots,$$

etc. Now apply these relations on both sides of the desired map to push the 2's and the 5 as far as possible to the right. By the complete symmetry we get

$$\left(J, [\underbrace{2,\cdots,2}_{p}]\right) + \cdots \rightarrow \left(J, [5, \underbrace{2,\cdots,2}_{k-1}]\right) + \cdots$$

to be shown. Thus we may obtain the generators of the form $[2,0,\cdots,0,2]$ by applying an admissible sequence to α_p, and their images satisfy (b).

For (c), consider the expression from (b),

$$[2,0,\cdots,0,2] \rightarrow [5,0,\cdots,0,2].$$

The first terms on each side are

$$(2i+2, i, \cdots) \rightarrow (2i+5, i, \cdots),$$

to this apply $(2(2i+2))$. We have

$$Sq^{2p}Sq^{p+3} = Sq^{2p+3}Sq^{p} + \text{higher terms}$$

so

$$\begin{aligned}(2(2i+2), 2i+2, i, \cdots) &\to (2(2i+2), (2i+2)+3, i, \cdots),\\ &= (2(2i+2)+3, 2i+2, i, \cdots) + \cdots.\end{aligned}$$

Continuing in this manner, we get

$$[0, \cdots, 0, 2, 0, \cdots, 0, 2] \to [3, 0, \cdots, 0, 2, 0, \cdots, 0, 2].$$

For (d), consider

$$[2, 0, \cdots, 0, 2] \to [5, 0, \cdots, 0, 2]$$

and to

$$(2i+2, i, \cdots) \to (2i+5, i, \cdots)$$

apply $(2(2i+2)+4)$. Since $Sq^{2p+4}Sq^{p+3} = Sq^{2p+5}Sq^{p+2} +$ higher terms,

$$\begin{aligned}(2(2i+2)+4, 2i+2, i, \cdots) &\to (2(2i+2)+4, 2i+2+3, i, \cdots),\\ &= (2(2i+2)+5, 2i+2, i, \cdots) + \cdots,\\ &= (2(2i+4)+1, 2i+4, i, \cdots) + \cdots.\end{aligned}$$

Now $2(2i+2)+4 = 2(2i+4)$, so we now apply $4(2i+4) = 2\cdot 2(2i+4)$ and apply the relation

$$Sq^{2p}Sq^{p+1} = Sq^{2p+1}Sq^{p}$$

to complete part (d).

For part (e) begin with the relation from (c)

$$[0, \cdots, 0, 2, \cdots, 2] \to [3, 0, \cdots, 0, 2, \cdots, 2].$$

The last terms are

$$(2i, i, \cdots) \to (2i+3, i, \cdots).$$

To this apply $(4i+4)$ and use

$$Sq^{2p}Sq^{p+1} = Sq^{2p+1}Sq^{p}$$

to get

$$\begin{aligned}(4i+4, 2i, i, \cdots) &\to (2(2i+2), (2i+2)+1, \ i, \cdots),\\ &= (2(2i+2)+1, 2i+2, i, \cdots).\end{aligned}$$

This clearly continues to give

$$[0, \cdots, 0, 4, 0, \cdots, 2, \cdots] \to [1, 0, \cdots, 0 2, 0, \cdots, 2, \cdots].$$

Finally, (a) and (f) are just the special cases when there are no 2's.

Thus we may assume α_k and all our basis elements of rank k and their images known for $k \leqq p$, and must show the existence of α_{p+1}.

Consider $(2^{p+2}+1)\alpha_p \xrightarrow{\tilde{S}q^3} (2^{p+2}+1, 2^{p+1}+1)\alpha_{p-1} = 0$, since $(2^{p+2}+1, 2^{p+1}+1) = 0$ by the Adem relations. By exactness of $\mathscr{A}/\mathscr{A}Sq^1 \xrightarrow{\tilde{S}q^3} \mathscr{A}/\mathscr{A}Sq^1 \xrightarrow{\tilde{S}q^3} \mathscr{A}/\mathscr{A}Sq^1$, this means $(\tilde{S}q^3)^{-1}((2^{p+2}+1)\alpha_p)$ is not empty. Let $Q = I + \sum J$, $J > I$, be an element of $(\tilde{S}q^3)^{-}((2^{p+2}+1)\alpha_p)$ which has I largest in lexicographic order among all such elements. We shall show that $I = (2^{\ \ +2}-2,\cdots,2^2-2)$, and hence that Q may be taken to be α_{p+1}.

Suppose $I \neq (2^{p+2}-2,\cdots,2^2-2)$. These two elements have the same degree, since $(2^{p+2}+1)\alpha_p = (2^{p+2}+1, 2^{p+1}-2,\cdots,2^2-2)+\cdots$. Thus I does not have rank $p+1$, since $(2^{\ \ +2}-2,\cdots,2^2-2)$ is the only element of this degree with rank $p+1$. Since there are no elements of rank p of this degree, rank $(I) \leqq p-1$. If I is not the smallest term of one of our basis elements, then by (a)–(f), $I + \sum K$, $K > I$, is in the image of $\tilde{S}q^3$. (Note: here we need that the map $\tilde{S}q^3$ is known on all basis elements of rank $\leqq p$ only.) Thus $\sum J + \sum K$ maps into $(2^{p+2}+1)\alpha_p$ and is larger than Q, contradicting our choice of Q. Thus I is the smallest term of one of our basis elements.

Now we note that $(\mathscr{A}/\mathscr{A}Sq^1)^{\deg Q}$ has as basis our basis vectors (of rank $\leqq p-1$), the images under $\tilde{S}q^3$ of our basis vectors (of rank $\leqq p$), and $(2^{p+2}-2,\alpha_p)$, by considering the least terms of each of these.

Write

$$Q = (I + \textstyle\sum K) + \sum\alpha + \sum\beta + \begin{cases} 0 \\ (2^{p+2}-2,\alpha_p), \end{cases}$$

with $I + \sum K$, α in our basis and β in the image under $\tilde{S}q^3$ of our basis. The term $(2^{p+2}-2,\alpha_p)$ must occur, since if not $\tilde{S}q^3(Q)$ has as its lowest term one of the images of our basis vectors, none of which is $[5,2,\cdots,2]$. Since this expression is formable by subtracting from Q the basis vectors in their lexicographic order, $I < (2^{p+2}-2,\cdots,2^2-2)$.

Now

$$\begin{aligned} Q + (2^{p+2}-2,\alpha_p) \xrightarrow{\tilde{S}q^3} & (2^{p+2}+1,\alpha_p) + (2^{p+2}-2, 2^{p+1}+1, \alpha_{p-1}), \\ &= (2^{p+2}+1,\alpha_p) + (2^{p+2}+1, 2^{p+1}-2,\alpha_{p-1}) \\ &\quad + (2^{p+2}, 2^{p+1}-1, \alpha_{p-1}) \end{aligned}$$

is an image of our basis vectors, namely $(I + \sum K) + \sum\alpha$. Taking the least terms of the right side lexicographically, begin to find the basis vectors which map to this image.

In our basis, $I' + \sum J' \to I'' + \sum J''$, with $i_1(I'')$ even implies by (a)–(f) that

$I_0' \neq \phi$. Thus the terms in $\tilde{S}q^3(Q + (2^{p+2}-2, \alpha_p))$ having $i_1 = 2^{p+2}$ come from elements $I' + \sum J'$ with $i_1(I') = 2^{p+2}$, which are larger than $(2^{p+2}-2,\cdots,2^2-2)$. Since all other terms in $\tilde{S}q^3(Q + (2^{p+2}-2, \alpha_p))$ have $i_1 \geqq 2^{p+2}+1$, we must have $I + \sum K \to L + \sum M$ with $i_1(L) \geqq 2^{p+2}+1$. Since $I < (2^{p+2}-2, \cdots, 2^2-2)$, $i_1(I) \leqq 2^{p+2}-2$. Thus under $\tilde{S}q^3$ the leading coefficient of I increases by 3 or more. From our calculations, the increase is always $\leqq 3$, and is 3 only if the element has leading term $[0,\cdots,0,2,\cdots,2]$.

Thus $I = [0,\cdots,0,2,0,\cdots,0,2,\cdots,2]$ with $\leqq p-1$ 2's. This is impossible, since the coefficient i_1 in such a sequence is $2^{k_1} + \cdots + 2^{k_n}$, where n is the number of 2's, and $i_1(I) = 2^{p+2} - 2 = 2^{p+1} + \cdots + 2$ has p 2's.

Thus $I = (2^{p+2}-2,\cdots,2^2-2)$, which completes the proof.

Corollary. *There exists an epimorphism*

$$f : \tilde{S}q^3(H^*(K(Z,2p))) \otimes Z_2[\psi_{2i} \mid L(2i) = p+1] \to H^*(K(Z,2p))/I(Sq^3 i_{2p})$$

such that the composition $\tilde{S}q^3 \cdot f$ is an isomorphism of the first factor of the tensor product with the image of $\tilde{S}q^3$ and is zero on the second factor.

Proof. $H^*(K(Z,2p))/I(Sq^3 i_{2p})$ is a quotient of the polynomial algebra on classes $Sq^{I+\Sigma J} i_{2p}$, where $I + \sum J$ belongs to our basis of $\mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^3$, and the excess of I is less than $2p$. Consider the map from this polynomial algebra into $H^*(K(Z,2p-3))$ by $\tilde{S}q^3$.

If $I = (I_0, I_1)$ with $e(I) < 2p$, $I_0 \neq \phi$, then $\tilde{S}q^3(I + \sum J) = I' + \sum J'$, and $i_1(I_0') = i_1(I_0) = i_1$ so $e(I') = 2i_1 - \deg I' = 2i_1 - \deg I - 3 = e(I) - 3 < 2p - 3$.

If I_0 is empty:

$I_1 = [0,\cdots,0,2,0,\cdots,0,2,\cdots,2]$ has excess $< 2p$ only if I_1 has less than p 2's, and $e(I') = e(I) + 3$, so $e(I) < 2p$, $e(I') \geqq 2p-3$ implies $e(I) = 2p-6, 2p-4, 2p-2$. If $e(I) = 2p-6$, $e(I') = 2p-3$.

$$I_1 = [0,\cdots,0,4,\underbrace{0,\cdots,0}_{q},2,\cdots,2,\cdots,2,2]$$

has excess $4 +$ number of 2's so is less than $2p$ only if there are less than $p-2$ 2's. If $q = 0$, $e(I') = e(I) - 1$, and if $q > 0$, $e(I') = e(I) - 1$. Thus $e(I') \geqq 2p-3$, $e(I) < 2p$ implies $e(I) = 2(p-3) + 4$, $e(I') = 2p-3$.

Now the sequences $[0,\cdots,0,2,\cdots,2]$ and $[0,\cdots,0,4,0,\cdots,0,2,\cdots,2]$ with $p-3$ 2's give images with excess $2p-3$ which thus have the form $(Sq^L i_{2p-3})^2$, where $L = I'' + \sum J''$,

$$\begin{aligned} I'' &= [0,\cdots,0,2,\cdots,2] \quad \text{with } p-4, p-3, \text{ or } p-2 \text{ 2's; or} \\ &= [0,\cdots,0,4,0,\cdots,2,\cdots,2] \quad \text{with } p-4 \text{ 2's} \end{aligned}$$

in cases b, c, e and d, respectively. Since these elements are not in our previous image, we see that our polynomial algebra maps to $H^*(K(Z, 2p-3))$ with kernel the polynomial ideal generated by

$$Z_2[Sq^{[0,\cdots,0,2,\cdots,2,\cdots,2]+\Sigma J} i_{2p} \mid p-1 \text{ or } p-2 \text{ 2's}].$$

Now consider sequences $[0,\cdots,0,2,\cdots,2,\cdots,2]$ with k 2's. This clearly divides into the termwise sum of k sequences $[0,\cdots,0,2]$ of distinct lengths, and these have degree $2+4+\cdots+2^r = 2^{r+1}-2$. Thus the degree of $[0,\cdots,0,2,\cdots,2,\cdots,2]$ is $2^{t_1}+\cdots+2^{t_k}-2k$ with $2 \leqq t_1 < \cdots < t_k$.

Now a sequence $I = [0,\cdots,0,2,\cdots,2]$ with $p-1$ 2's when applied to a class of dimension $2p$ has dimension $2^{t_1}+\cdots+2^{t_{p-1}}-2(p-1)+2p = 2+2^{t_1}+\cdots+2^{t_{p-1}}$ L of this is $p+1$.

For a sequence $I = [0,\cdots,0,2,\cdots,2]$ with $p-2$ 2's when applied to a class of dimension $2p$, we have dimension $2^{t_1}+\cdots+2^{t_{p-2}}-2(p-2)+2p = 4+2^{t_1}$ $+\cdots+2^{t_{p-2}}$, so if i is this dimension, $i-1 = 1+2+2^{t_1}+\cdots+2^{t_{p-2}}$ or $L(i) = p+1$.

In §3, we will see that this epimorphism is in fact an isomphism. This will show that $H^*(K(Z,2p))/I(Sq^3 i_{2p})$ is a polynomial algebra. The situation is more difficult for $H^*(K(Z,2p+1))/I(Sq^3 i_{2p+1})$ for we have seen that

$$(Sq^L i_{2p+4-3})^2 \equiv 0 \mod I(Sq^3 i_{2p+1}),$$

while $Sq^L i_{2p+4-3} \not\equiv 0$.

Proposition. *There exist epimorphisms*

$$\tilde{S}q^5(H^*(K(Z,8k))) \otimes \frac{H^*(K(Z_2,2^{4k-1}))}{I(Sq^1 i,\cdots,Sq^{2^{4k-3}} i)} \to H^*(K(Z,8k))/I(Sq^2 i_{8k}),$$

$$\tilde{S}q^2(H^*(K(Z_2,8k+1))) \otimes \frac{H^*(K(Z_2,2^{4k}))}{I(Sq^1 i,\cdots,Sq^{2^{4k-2}} i)} \to H^*(K(Z_2,8k+1))\ I(Sq^2 i_{8k+1}),$$

$$\tilde{S}q^2(H^*(K(Z_2,8k+2))) \otimes \frac{H^*(K(Z_2,2^{4k+1}))}{I(Sq^1 i,\cdots,Sq^{2^{4k-1}} i)} \to H^*(K(Z_2,8k+2))/I(Sq^3 i_{8k+2})$$

$$\tilde{S}q^3(H^*(K(Z,8k+4))) \otimes \frac{H^*(K(Z_2,2^{4k+2}))}{I(Sq^1 i,\cdots,Sq^{2^{4k}} i)} \to H^*(K(Z,8k+4))/I(Sq^5 i_{8k+4})$$

such that the compositions with $\tilde{S}q^j$ ($j = 5,2,2,3$ *respectively*) *give an isomorphism of the first factor with the image of* $\tilde{S}q^j$ *and are zero on the second factor.*

The proof of this proposition is nearly identical to the proof of the preceding proposition and corollary. This result is based on the exact rectangle of Toda [6] and uses the same method of ordering the Cartan basis lexicographically. The important point is the existence of elements

$$\alpha_k = (2^k - 4, 2^{k-1} - 2, 2^{k-2} - 1, 2^{k-3} - 1, 2^{k-4} - 4, \cdots) + \cdots$$

and

$$\alpha'_k = (2^{k-1} - 2, 2^{k-2} - 1, 2^{k-3} - 1, 2^{k-4} - 4, \cdots) + \cdots$$

such that $\tilde{S}q^i(\alpha_k) = (2^{k+1} + 1, \alpha_{k-1})$ and $\tilde{S}q^i(\alpha'_k) = (2^{k-1} + 1, \alpha'_{k-1})$. Here $\alpha_{4j}, \alpha'_{4j} \in \mathscr{A}/\mathscr{A}Sq^3$; $\alpha_{4j+1}, \alpha'_{4j+1} \in \mathscr{A}/\mathscr{A}Sq^1 + \mathscr{A}Sq^5$; $\alpha_{4j+2}, \alpha'_{4j+2} \in \mathscr{A}/\mathscr{A}Sq + \mathscr{A}Sq^2$; and $\alpha_{4j+3}, \alpha'_{4j+3} \in \mathscr{A}/\mathscr{A}Sq^2$.

3. $H^*(BO(k, \cdots, \infty))$ **and** $H^*(BU(k, \cdots, \infty))$. By the Bott periodicity results, we have

i (modulo 8)	0	1	2	3	4	5	6	7
$\pi_i(BO)$	Z	Z_2	Z_2	0	Z	0	0	0

and

$$\pi_i(BU) = \begin{cases} Z & i \text{ even, for } i > 0. \\ 0 & i \text{ odd,} \end{cases}$$

Thus $BO(k, \cdots, \infty) = BO(k+1, \cdots, \infty)$ if $k \not\equiv 0,1,2,4$ (8) and $BU(2p-1, \cdots, \infty) = BU(2p, \cdots, \infty)$.

Let $\phi(0,k)$ be the number of integers s such that $0 < s \leqq k$ and such that $s \equiv 0,1,2,4$ (modulo 8).

Let $\pi_k = \pi_k(BO)$.

We wish to prove:

THEOREM A. *If* $k \equiv 0,1,2,4$ (*modulo* 8):

(a) $H^*(BO(k, \cdots, \infty)) \cong H^*(K(\pi_k, k))/I(Q_k i_k) \otimes Z_2[\theta_i \mid L(i) > \phi(0,k)]$ *where*

$$Q_k = \begin{cases} Sq^2 & \text{if } k \equiv 0,1 \ (8), \\ Sq^3 & \text{if } k \equiv 2 \ (8), \\ Sq^5 & \text{if } k \equiv 4 \ (8). \end{cases}$$

(b) *The spectral sequence with fiber* $H^*(K(\pi_k, k-1))$, *base* $H^*(K(\pi_k,k))\ I(Q_k i_k)$, *and* i_{k-1} *transgressive to* i_k *(deduced from the spectral sequence of the fibration* $K(\pi_k,k) \rightarrow PK(\pi_k,k) \rightarrow K(\pi_k,k)$ *by taking quotients) has as its* E^∞ *term only those elements of the fiber belonging to the polynomial subalgebra generated by all* $Sq^I(Q_k i_{k-1})$ $(k > 1)$.

THEOREM B.

(a) $H^*(BU(2p, \cdots, \infty)) \cong H^*(K(Z,2p))/I(Sq^3 i_{2p}) \otimes Z_2[\theta_{2i} \mid L(2i) > p+1]$.

(b) *The spectral sequence with fiber* $H^*(K(Z,2p-1))$, *base* $H^*(K(Z,2p))/I(Sq^3 i_{2p})$ *and* i_{2p-1} *transgressive to* i_{2p} *has as its* E^∞ *term only those elements of the fiber* *belonging to the polynomial subalgebra generated by all* $Sq^I Sq^3 i_{2p-1}$ $(p \geqq 1)$.

Proof of Theorem A. The proof is by induction on k.

For $k=1$, $H^*(BO(1,\cdots,\infty)) = H^*(BO) = Z_2[\theta_i]$. But $\phi(0,1) = 1$, so $Z_2[\theta_i] \cong Z_2[\theta_1] \otimes Z_2[\theta_i | L(i) > 1]$ and $Z_2[\theta_1] \cong H^*(K(Z_2,1))$, which completes this case.

For $k=2$, $BO(2,\cdots,\infty) = BSO$ and is well as known,

$$\begin{aligned} H^*(BSO) &\cong Z_2[w_i | i > 1] \cong Z_2[\theta_i | L(i) > 1] \\ &\cong Z_2[\theta_i | L(i) = 2] \otimes Z_2[\theta_i | L(i) > \phi(0,2) = 2]. \end{aligned}$$

Now the only admissible sequences of excess < 2 are $I = (\phi)$ and $I = (2^k,\cdots,1)$, which are Θ^1-sequences, so

$$Z_2[\theta_i | L(i) = 2] \cong H^*(K(Z_2,2)) \cong H^*(K(Z_2,2))/I(Sq^1 i_2).$$

Also, for $k=2$, the spectral sequence is just the spectral sequence of $K(Z_2,1) \to PK(Z_2,2) \to K(Z_2,2)$ which has $E^\infty = 0$.

Now suppose $k \equiv 0,1,2,4$ (8), $k > 2$, and that the theorem is true for all $u \equiv 0,1,2,4$ (8) with $u < k$. In particular, choose $j \equiv 0,1,2,4$ (8), $j < k$, such that if $j < v < k$ then $v \not\equiv 0,1,2,4(8)$.

Thus we have the fibration

$$K(\pi_j, j-1) \to BO(k,\cdots,\infty) \to BO(j,\cdots,\infty)$$

which is induced from

$$K(\pi_j, j-1) \to PK(\pi_j, j) \to K(\pi_j, j).$$

Thus the spectral sequence of $BO(k,\cdots,\infty) \to BO(j,\cdots,\infty)$ has fiber $H^*(K(\pi_j, j-1))$, base $H^*(BO(j,\cdots,\infty)) \cong H^*(K(\pi_j,j))/I(Q_j i_j) \otimes Z_2[\theta_i | L(i) > \phi(0,j)]$, and i_{j-1} transgresses to $i_j \otimes 1$. This is the tensor product of the spectral sequence in $(b)_j$ with the algebra $Z_2[\theta_i | L(i) > \phi(0,j)]$, so by $(b)_j$, E^∞ of this spectral sequence is $(PQ_j i_{j-1}) \otimes Z_2[\theta_i | L(i) > \phi(0,j)]$ where $(PQ_j i_{j-1})$ denotes the polynomial subalgebra of $H^*(K(\pi_j, j-1))$ generated by all $Sq^I Q_j i_{j-1}$.

Thus $H^*(BO(k,\cdots,\infty)) \cong (PQ_j i_{j-1}) \otimes Z_2[\theta_i | L(i) > \phi(0,j)]$. The class of dimension k in $H^*(BO(k,\cdots,\infty))$ is $Q_j i_{j-1}$ if $j > 4$ and θ_k if $j \leqq 4$.

If $j > 4$, $BO(k,\cdots,\infty) \to K(\pi_k, k)$ gives

$$\begin{array}{ccccc} H^*(K(\pi_k,k)) & \to & H^*(BO(k,\cdots,\infty)) & \to & H^*(K(\pi_j,j-1)) \\ i_k & & \longrightarrow & & Q_j i_{j-1}. \end{array}$$

By the exact rectangle of Toda [6] used in §2, we have $Q_k i_k \to 0$. Further, if $j > 4$, $2^{\phi(0,j)} > k + \deg Q_k$, so that $Q_k i_k$ projects to zero in the second factor of $H^*(BO(k,\cdots,\infty))$ and hence $Q_k i_k$ goes to zero in $H^*(BO(k,\cdots,\infty))$.

Thus

$$H^*(K(\pi_k, k))/I(Q_k i_k) \to H^*(BO(k, \cdots, \infty)),$$

mapping $i_k \to Q_j i_{j-1} \otimes 1$.

Now consider the bundle induced by $BO(k, \cdots, \infty) \to BO$ from the universal vector bundle and form the Thom space $TBO(k, \cdots, \infty)$. Letting U be the basic class, we have

$$Sq^i U = 0 \quad \text{if } 1 \leqq i < 2^{\phi(0,j)}$$

$$Sq^{2^{\phi(0,j)}} U = w_{2^{\phi(0,j)}} \cdot U = [1 \times \theta_{2^{\phi(0,j)}}] \cdot U.$$

For $j > 4$, $2^{\phi(0,j)} \geqq 16$ and by Adams [2], there are secondary cohomology operations Φ_λ, and primary cohomology operations P_λ such that

$$\sum P_\lambda(\Phi_\lambda U) = Sq^{2^{\phi(0,j)}} U.$$

Further, the secondary operations $\Phi_\lambda U \in H^\lambda(TBO(k, \cdots, \infty))/J$ and $P_\lambda(J) = 0$. Thus picking classes $x_\lambda \cdot U$ in $H^\lambda(TBO(k, \cdots, \infty))$ representing $\Phi_\lambda U$, we have

$$\sum P_\lambda(x_\lambda \cdot U) = Sq^{2^{\phi(0,j)}} U.$$

Now degree $P_\lambda < 2^{\phi(0,j)}$, so expanding by the Cartan formula,

$$\sum P_\lambda(x_\lambda \cdot U) = \sum (P_\lambda x_\lambda) \cdot U.$$

Now degree $P_\lambda > 0$, so $\dim x_\lambda < 2^{\phi(0,j)}$, and so x_λ is in the image of a class $\alpha_\lambda \in H^\lambda(K(\pi_k,k))$. Thus $\alpha = \sum P_\lambda \alpha_k \in H^{2^{\phi(0,j)}}(K(\pi_k,k))$ maps to $w_{2^{\phi(0,j)}} = 1 \otimes \theta_{2^{\phi(0,j)}}$ in $H^*(BO(k, \cdots, \infty))$.

If $j \leqq 4$, $\alpha = i_k$ maps to $1 \otimes \theta_{2^{\phi(0,j)}}$.

Thus

$$H^*(BO(k, \cdots, \infty)) \to (PQ_j i_{j-1}) \otimes Z_2[\theta_1 | L(i) = \phi(0,j) + 1]$$

maps $H^*(K(\pi_k,k))/I(Q_k i_k)$ onto the latter. Thus by the results of §2, we have epimorphisms

$$\tilde{Q}_j(H^*(K(\pi_k,k))) \otimes Z_2[\theta_i \,|\, L(i) = \phi(0,k)] \xrightarrow{f} H^*(K(\pi_k,k))/I(Q_k i_k)$$

and

$$H^*(K(\pi_k,k))/I(Q_k i_k) \xrightarrow{g} (PQ_j i_{j-1}) \otimes Z_2[\theta_i \,|\, L(i) = \phi(0,j) + 1],$$

but $\tilde{Q}_j(H^*(K(\pi_k,k))) = (PQ_j i_{j-1})$ and $\phi(0,j) + 1 = \phi(0,k)$, and by considering the mappings, $g \circ f$ is an isomorphism. Thus f and g are isomorphisms.

This implies that

$$H^*(K(\pi_k, k))/I(Q_k i_k) \otimes Z_2[\theta_i \,|\, L(i) > \phi(0,k)] \to H^*(BO(k, \cdots, \infty))$$

is an isomorphism.

Now consider $(b)_k$. Let $\mathscr{A}_0 = 0$ if $k \equiv 1,2$ (8), $\mathscr{A}_0 = \mathscr{A}Sq^1$ if $k \equiv 0,4$ (8). By the results in §2, $\mathscr{A}/\mathscr{A}_0$ has a basis consisting of our basis vectors, and the images of our basis vectors.

Let $B_q \subset \mathscr{A}/\mathscr{A}_0$ be the set of all our basis vectors of degree q, $I + \sum J$, with the excess of I less than k. Let $C_q \subset \mathscr{A}/\mathscr{A}_0$ be the set of all images $I' + \sum J'$ of our basis vectors, with excess of I' less than k.

Now for each $I + \sum J$ in B_q, define a spectral sequence ${}_IE$ by

$$ {}_IE^{r,s} = \begin{cases} 0 & \text{if } r \neq 0,\ q+k-1 \text{ or } s \not\equiv 0 \text{ (modulo } q+k), \\ (Sq^{I+\Sigma J}\, i_{k-1}) \cdot (Sq^{I+\Sigma J}\, i_k)^p & \text{if } r = q+k-1, s = p(q+k), \\ (Sq^{I+\Sigma J}\, i_k)^p & \text{if } r = 0,\ s = p(q+k), \end{cases} $$

(r = fiber degree, s = base degree) and in which the only nonzero differentials are $d : {}_IE^{q+k-1,p(q+k)} \to {}_IE^{0,(p+1)(q+k)} : (Sq^{I+\Sigma J} i_{k-1})(Sq^{I+\Sigma J} i_k)^p \to (Sq^{I+\Sigma J} i_k)^{p+1}$.

For each $I' + \sum J' \in C_q$, define a spectral sequence ${}_{I'}E$ by

$$ {}_{I'}E^{r,s} = \begin{cases} 0 & (r,s) \neq (q+k-1,0) \text{ or } (0,0), \\ 1 & (r,s) = (0,0), \\ Sq^{I'+\Sigma J'} i_{k-1} & (r,s) = (q+k-1,0), \end{cases} $$

and all differentials are zero.

Now let E denote the spectral sequence of $(b)_k$. For each $I + \sum J \in B_q$, there is an obvious inclusion map ${}_IE \to E$, which commutes with differentials. Further, there is a map ${}_{I'}E \to E$ for all $I' + \sum J' \in C_q$ (since $Sq^{I'+\Sigma J'}\ i_{k-1}$ in E transgresses in the spectral sequence of the fibration $K(\pi_k, k-1) \to PK(\pi_k,k) \to K(\pi_k,k)$ to $Sq^{I'+\Sigma J'} i_k$, which belongs to $I(Q_k i_k)$. Note that in $K(\pi_k,k-1) \to PK(\pi_k,k) \to K(\pi_k,k)$ the elements $Sq^{I+\Sigma J}\, i_{k-1}$ and $Sq^{I'+\Sigma J'}\, i_{k-1}$ transgress to $Sq^{I+\Sigma J}\, i_k$ and $Sq^{I'+\Sigma J'}\, i_k$ modulo elements $Sq^K i_k$, with excess of K equal to k. However, in the spectral sequence of the fibration, $Sq^K i_k$ is in the image of a d^t for a lesser t, and so the transgressions are as asserted).

Thus by using the products in E, we have a map of spectral sequences

$$ \Big(\bigotimes_{\cup B} {}_IE \Big) \Big(\bigotimes_{\cup C_q} {}_{I'}E \xrightarrow{\rho} E \Big). $$

Now the elements of $(\bigcup_q B_q) \cup (\bigcup_q C_q)$ give a simple system of transgressive generators for $H^*(K(\pi_k, k-1))$ in the fibration, so ρ is an isomorphism (as vector spaces) on the fiber.

The base in

$$ \Big(\bigotimes_{\cup B_q} {}_IE \Big) \otimes \Big(\bigotimes_{\cup C_q} {}_{I'}E \Big) $$

is by definition the polynomial algebra over Z_2 on $\{Sq^{I+\Sigma J} i_k \mid I + \sum J \in B_{\deg I}\}$, which from §2 and $(a)_k$ is in fact $H^*(K(\pi_k,k))/I(Q_k i_k)$. Thus ρ is an isomorphism (as algebras) on the base.

Then

$$E^\infty \cong \left(\underset{\cup B_q}{\otimes} {}_I E^\infty \right) \otimes \left(\underset{\cup C_q}{\otimes} {}_{I'} E^\infty \right) \cong \underset{\cup C_q}{\otimes} {}_{I'} E^\infty \cong \underset{\cup C_q}{\otimes} {}_{I'} E$$

for the tensor products are finite in degree (r,s) for all (r,s). Thus E^∞ has as its only terms the elements of the fiber which belong to the polynomial algebra on $Sq^{I'+\Sigma J'} i_{k-1}\}$ with $I' + \sum J' \in C_q$. By the results of §2 and $(a)_k$ this is precisely $(PQ_k i_{k-1})$.

The proof of Theorem B can be easily seen to be formally identical except in the lowest dimensions. For these we have the following:

$H^*(BU(2, \cdots, \infty)) = H^*(BU) = Z_2[\theta_{2i}]$, and the map $BU \to K(Z,2)$ sends i_2 into θ_2. $H^*(K(Z,2))$ is the Z_2 polynomial algebra on all $Sq^I i_2$ with I admissible, $e(I) < 2$, $i_r(I) > 1$. There are no such sequences, so

$$H^*(K(Z,2)) = H^*(K(Z,2))/I(Sq^3 i_2) \cong Z_2[i_2] \cong Z_2[\theta_2].$$

The spectral sequence then gives $H^*(BU(4, \cdots, \infty)) = Z_2[\theta_{2i} \mid L(2i) > 2]$. Now $H^*(K(Z,4))$ is the Z_2 polynomial algebra on all $Sq^I i_4$ with I admissible, $e(I) < 4$, $i_r(I) > 1$, so

$$I = (2^k 3, \cdots, 3),\ (2^k, \cdots, 2), (2^j(2^{k+1} + 1), \cdots, 2^{k+1} + 1, 2^k, \cdots, 2)$$
$$= (2^j(2^{k+1} + 1), \cdots, 2(2^{k+1} + 1), 2^{k+1}, \cdots, 4, 3),$$

so

$$H^*(K(Z;4))/I(Sq^3 i_4) \cong Z_2[Sq^{(2^k, \cdots, 2)} i_4] \cong Z_2[\theta_{2i} \mid L(2i) = 3].$$

The general argument then gives $H^*(BU(6, \cdots, \infty)) \cong (PSq^3 i_3) \otimes Z_2[\theta_{2i} \mid L(2i) > 3]$ and the problem becomes to consider the map

$$H^*(K(Z,6)) \to H^*(BU(6, \cdots, \infty)) \to (PSq^3 i_3) \otimes Z_2[\theta_{2i} \mid L(2i) = 4],$$

in which $i_6 \to Sq^3 i_3 \otimes 1 = i_3^2 \otimes 1$. We must show that $Sq^2 i_6 \to 1 \otimes \theta_8$ in order to show that this is epic. The map $BU(6, \cdots, \infty) \to K(Z,6)$ induces the fibration $K(Z,5) \to BU(8, \cdots, \infty) \to BU(6, \cdots, \infty)$ and in the spectral sequence of this fibration, the transgression (either 0 or $1 \otimes \theta_8$) of $Sq^2 i_5$ is the image of $Sq^2 i_6$. However, if the transgression of $Sq^2 i_5$ is zero, then $Sq^2 i_5$ lasts to E^∞, so $H^7(BU(8, \cdots, \infty)) \neq 0$. Since $BU(8, \cdots, \infty)$ is 7-connected, this is impossible. Thus $Sq^2 i_6 \to 1 \otimes \theta_8$.

The remainder of the proof of the theorem then follows the general pattern as given for $BO(k, \cdots, \infty)$.

Corollary. *$H^*(BO(k, \cdots, \infty))$, $H^*(BU(k, \cdots, \infty))$, $H^*(K(Z,2p))/I(Sq^3 i_{2p})$, and $H^*(K(\pi_k, k))/I(Q_k i_k)$ are all polynomial algebras over Z_2.*

NOTE. The only problems remaining for the determination of the action of the Steenrod algebra on $H^*(BO(k,\cdots,\infty))$ and $H^*(BU(k,\cdots,\infty))$ are the relations for the θ's and the relation between $H^*(K(\pi_k,k))$ and the θ's. This last is given by knowing the classes $\bar{\alpha}_k \in H^{2^{\phi(0,k)}-1}(K(\pi_k,k);Z_2)$ and $\bar{\beta}_p \in H^{2^p}(K(Z,2p);Z_2)$ which map into $\theta_{2^{\phi(0,k)}-1}$ and θ_{2^p} respectively. These are in fact the elements $Sq^{\alpha_q}i$, where α_q were the elements found in §2.

One can show that

$$\begin{aligned}\bar{\alpha}_k &= i_k \quad \text{for } k = 1,2,4,8,\\ \bar{\alpha}_9 &= (Sq^4Sq^2Sq^1 + Sq^7)\,i_9,\\ \bar{\alpha}_{10} &= (Sq^{12}Sq^6Sq^3Sq^1 + Sq^{14}Sq^6Sq^2 + Sq^{15}Sq^4Sq^2Sq^1 + Sq^{15}Sq^{'}\\ &\qquad + Sq^{16}Sq^4Sq^2)i_{10},\end{aligned}$$

and

$$\begin{aligned}\bar{\beta}_p &= i_{2p} \quad \text{for } p = 1,2,\\ \bar{\beta}_3 &= Sq^2 i_6,\\ \bar{\beta}_4 &= (Sq^6Sq^2 + Sq^8)i_8.\end{aligned}$$

4. Applications.

DEFINITION. A manifold M is k-parallelizable if for every complex K of dimension $\leqq k$, and for every map $f: K \to M$, the bundle induced from the tangent bundle of M is trivial.

THEOREM. *A k-parallelizable differentiable manifold of dimension less than* $2^{\phi(0,k)+1}$ *is cobordic to zero (in the unoriented sense).*

Proof. Let $j \leqq k$ with $j \equiv 0,1,2,4$ (8) and such that there is no integer u with $j < u \leqq k$ and $u \equiv 0,1,2,4$ (8). Then $\phi(0,j) = \phi(0,k)$ and so we may assume $k \equiv 0,1,2,4$ (8).

Since M^n is k-parallelizable, the classifying map $\tau: M^n \to BO$ for the tangent bundle induces the zero map in homotopy in dimensions less than or equal to k. Thus τ lifts to a map $\hat{\tau}: M^n \to BO(k,\cdots,\infty)$ and in the induced map on cohomology, the class of dimension k goes to zero.

Thus $\tau^*(H^*(BO)) = \hat{\tau}^*(Z_2[\theta_i \mid L(i) > \phi(0,k)])$, or $w_i(M^n) = 0$ for $i < 2^{\phi(0,k)}$. Thus $v_i(M^n) = \tau^*(v_i) = 0$ for $i < 2^{\phi(0,k)}$, where v_i are defined by the equations $w_p = \sum_{r=0}^{p} Sq^{p-r}v_r$. Since $n < 2^{\phi(0,k)+1}$, $n/2 < 2^{\phi(0,k)}$ and $v_i(M^n) = 0$ for $i > n/2$ or $i \geqq 2^{\phi(0,k)}$.

Then $v_i(M^n) = 0$ if $i > 0$ or $w_i(M^n) = 0$ for all $i > 0$. Thus all Whitney classes and numbers of M^n are zero so M^n is cobordic to zero.

DEFINITION. A manifold M^n is weakly complex if the structure group of the bundle of M^n is reducible to the unitary group.

THEOREM. *A k-parallelizable weakly complex differentiable manifold of dimension less than* $2^{[k/2]+2}$ *is cobordic to zero (in the unoriented sense).*

Proof. Being weakly complex, the classifying map for the tangent bundle of M^n lifts to BU, and then to $BU(2[k/2], \cdots, \infty)$ by k-parallelizability. Further, the class of dimension $2[k/2]$ in cohomology goes to zero in M^n. Thus $\tau^*(H^*(BO)) = \tau^*(H^*(BU)) = \hat{\tau}^*(Z_2[\theta_{2i} | L(2i) > [k/2] + 1])$, or $w_i(M^n) = 0$ for $i < 2^{[k/2]+1}$. As in the preceding theorem M^n is then cobordic to zero for $n < 2^{[k/2]+2}$.

NOTE. $[k/2] + 2 \geqq \phi(0, k) + 1$, and $[k/2] + 2 > \phi(0, k) + 1$ for $k \equiv 0, 6, 7$ (modulo 8).

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